same cannot be said for the manipulations required in carrying out the variational procedure.

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Method for Constructing a Full Modal Damping Matrix from Experimental Measurements

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THE normal mode method is widely used for dynamic analysis of linear structures. By enabling the equations of motion to be written in terms of modal coordinates, solutions are more readily determined. Fortunately, structural damping tends to be small so that the classical undamped modes have a useful physical interpretation. It is common practice to introduce damping only after the equations have been transformed to modal coordinates. In this case, viscous damping is typically assumed and the modal damping matrix is taken to be diagonal. The elements along the diagonal are related to the percent of critical damping for each mode, while the rest of the matrix is assumed to be null implying that the modes are not coupled by damping forces in the structure.

If linear viscous damping is assumed at the outset, there are in fact only certain special cases for which the damping matrix is diagonalized by a transformation to modal coordinates. One case is that of proportional damping where the damping matrix is a linear combination of the mass and stiffness matrices.[1] The complementary class is referred to as nonproportional damping which in general does not diagonalize, although Caughey has identified subsets of this class which do.[2] Nevertheless, there seems to be no strong physical basis for any of the special cases so that in all probability the off-diagonal coupling terms will exist. In general they will be of the same order as the diagonal terms as will subsequently be shown.

Although justification may be found for neglecting these terms in some analyses, there are certainly times when this is inappropriate. For example when modal synthesis is employed to combine substructure characteristics in deriving the equations of motion for a complete structure, and linear viscous damping is taken to represent the dissipative mechanism of the structure, the full modal damping matrices are required for each substructure. Since the off-diagonal terms are likely to be of the same order as the diagonal ones, they too will influence the modal damping being computed for the complete structure. Although objections have been raised over the use of viscous damping,[3] it appears doubtful that the use of more general damping models can be justified yet, since considerably more information is required to define their parameters. The proper use of viscous damping should be fully explored first since it represents the simplest approach. A method for measuring the off-diagonal terms of the modal damping matrix is the subject of this Note. Since this entails more work, the likelihood of proportional damping should first be established.

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A Necessary Condition for the Existence of Proportional Damping

If proportional damping is presumed to exist, the homogeneous equations of motion can be written in the form

$$m\ddot{x}+(\alpha m+\beta k)\dot{x}+kx = 0 \tag{1}$$

where x is a vector of generalized displacements, m and k are, respectively, the mass and stiffness matrices of the structure, and α and β are scalar constants. Solutions will be of the form

$$x(t) = \phi_{R_j} e^{\lambda_j t} = \phi_{R_j} e^{(\sigma_j + i\omega_j)t} \tag{2}$$

where ϕ_{R_j} denotes the real modal displacement vector of the jth mode and $\lambda_j = \sigma_j + i\omega_j$ is the corresponding complex eigenvalue. The quantities ω_j and $-\sigma_j$ are interpreted as the damped modal frequency and decay rate. The undamped frequency ω_{0_j} and critical damping ratio ζ_j are given in terms of σ_j and ω_j by

$$\omega_{0_j} = (\sigma_j^2+\omega_j^2)^{1/2}, \quad \zeta_j = -\sigma_j/(\sigma_j^2+\omega_j^2)^{1/2}$$

In the case of proportional damping, it may be shown that ζ_j and ω_{0_j} for each mode are related by

$$\zeta_j = \alpha/\omega_{0_j} + \beta\omega_{0_j} \tag{3}$$

Regardless of whether proportional damping exists or not, both σ_j and ω_j can be measured. However, rather large uncertainties are usually associated with measurements of σ_j which in turn reflect upon ζ_j. Corresponding pairs of ζ_j and ω_{0_j} can be plotted as in Fig. 1 with uncertainty bands indicated by the vertical line segments. If proportional damping exists, a curve corresponding to Eq. (3) could be passed through all of the uncertainty bands as indicated by the dotted line in Fig. 1. Otherwise nonproportional damping would be concluded. Although this is interpreted as a necessary condition for establishing proportional damping, it is not a sufficient condition. If it were, one would be led to conclude, for example, that all two degree-of-freedom systems have proportional damping since there are two arbitrary constants in Eq. (3) and such a curve can be fitted through any two points. Thus it should be possible to establish quite easily when nonproportional damping is present.

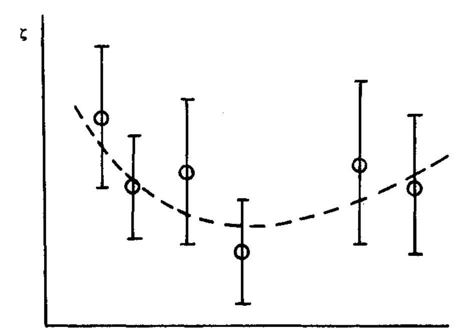

Fig. 1 A necessary condition for proportional damping.

A Perturbation Method for Calculating the Full Modal Damping Matrix

For linear systems with viscous damping, the n homogeneous equations of motion may be written

$$m\ddot{x}+c\dot{x}+kx = 0 \tag{4}$$

where c denotes the viscous damping matrix. Transformation of these equations to the undamped modal coordinates q results in

$$\phi_R{}^T m\phi_R\ddot{q}+\phi_R{}^T c\phi_R\dot{q}+\phi_R{}^T k\phi_R q = 0$$

or

$$M\ddot{q}+C\dot{q}+Kq = 0 \tag{5}$$

The matrices M and K will be diagonal while in general C

will not. However, Eq. (4) can be written in first order form as

$$\begin{bmatrix} c & m \\ m & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = 0 \qquad (6)$$

Solutions can be found of the form

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{Bmatrix} \phi_j \\ \phi_j \lambda_j \end{Bmatrix} e^{-\lambda_j t}$$

where λ_j and ϕ_j are the complex eigenvalues and eigenvectors from the $2n$ eigenproblem

$$\left(\begin{bmatrix} c & m \\ m & 0 \end{bmatrix}^{-1}\begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix} - \lambda_j \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right)\begin{Bmatrix} \phi_j \\ \phi_j \lambda_j \end{Bmatrix} = 0 \qquad (7)$$

as discussed in Ref. 1. Solution of this eigenproblem results in a set of coupled eigenvalues and eigenvectors which in the case of light damping may be written in matrix form as

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \phi & \phi^* \\ \phi\lambda & \phi^*\lambda^* \end{bmatrix}$$

Here λ denotes a diagonal matrix of complex eigenvalues and ϕ denotes the corresponding complex modal matrix. The asterisk denotes the complex conjugate. Since both of the square matrices in Eq. (6) are symmetric, the matrix

$$\begin{bmatrix} \phi^T & \lambda\phi^T \\ \phi^{*T} & \lambda^*\phi^{*T} \end{bmatrix}\begin{bmatrix} c & m \\ m & 0 \end{bmatrix}\begin{bmatrix} \phi & \phi^* \\ \phi\lambda & \phi^*\lambda^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is diagonal. In particular then

$$A_{12} = \phi^T c\phi^* + \phi^T m\phi^*\lambda^* + \lambda\phi^T m\phi^* = 0 \qquad (8)$$

At this point, a perturbation analysis can be formulated by writing ϕ and ϕ^* in the form

$$\phi = \phi_R + \delta\phi_R + i\delta\phi_I \qquad (9a)$$

$$\phi^* = \phi_R + \delta\phi_R - i\delta\phi_I \qquad (9b)$$

where the complex eigenvectors ϕ are expressed as the sum of the hypothetical undamped eigenvectors ϕ_R which are real, and the relatively small complex perturbation terms $\delta\phi_R + i\delta\phi_I$. Substitution of Eq. (9) into Eq. (8) and the use of $\lambda_j = \sigma_j + i\omega_j$ leads to

$$\begin{aligned} &\phi_j^T c\phi_k^* + \phi_j^T m\phi_k^*\lambda_k^* + \lambda_j\phi_j^T m\phi_k^* = \\ &\quad \phi_{R_j}^T c\phi_{R_k} + (\phi_{R_j}^T c\delta\phi_{R_k} + \delta\phi_{R_j}^T c\phi_{R_k}) \\ &\quad - i(\phi_{R_j}^T c\delta\phi_{I_k} - \delta\phi_{I_j}^T c\phi_{R_k}) + (\delta\phi_{R_j}^T c\delta\phi_{R_k} + \delta\phi_{I_j}^T c\delta\phi_{I_k}) \\ &\quad - i(\delta\phi_{R_j}^T c\delta\phi_{I_k} - \delta\phi_{I_j}^T c\delta\phi_{R_k}) + [(\sigma_j + \sigma_k) + i(\omega_j - \omega_k)] \times \\ &\quad [\phi_{R_j}^T m\phi_{R_k} + (\phi_{R_j}^T m\delta\phi_{R_k} + \delta\phi_{R_j}^T m\phi_{R_k}) \\ &\quad - i(\phi_{R_j}^T m\delta\phi_{I_k} - \delta\phi_{I_j} m\phi_{R_k}) + (\delta\phi_{R_j}^T m\delta\phi_{R_k} + \delta\phi_{I_j}^T m\delta\phi_{I_k}) \\ &\qquad - i(\delta\phi_{R_j}^T m\delta\phi_{I_k} - \delta\phi_{I_j}^T m\delta\phi_{R_k})] = 0 \end{aligned}$$

In the preceding equation, ω, m, and ϕ_R are treated as zeroth-order terms while σ, c, $\delta\phi_R$, and $\delta\phi_I$ are treated as first-order terms. Zeroth- and first-order product terms, real and imaginary, are then separately equated to zero. This leads to the following equations:

$$(\omega_j - \omega_k)\phi_{R_j}^T m\phi_{R_k} = 0 \qquad (10a)$$

$$\phi_{R_j}^T c\phi_{R_k} = -(\sigma_j + \sigma_k)\phi_{R_j}^T m\phi_{R_k}; \quad j = k \qquad (10b)$$

$$\phi_{R_j}^T c\phi_{R_k} = -(\omega_j - \omega_k)(\phi_{R_j}^T m\delta\phi_{I_k} - \phi_{R_j}^T m\phi_{R_k}); \quad j \neq k \qquad (10c)$$

Equation (10a) is simply a statement of the orthogonality of ϕ_R with respect to m. Equations (10b and c) become

$$C_{jj} = -2\sigma_j M_{jj} \qquad (11a)$$

$$\begin{aligned} C_{jk} &= -(\omega_j - \omega_k)(\phi_{R_j}^T m\delta\phi_{I_k} - \delta\phi_{I_j}^T m\phi_{R_k}) \\ &= \omega_j\delta\phi_{I_j}^T m\phi_{R_k} + \omega_k\phi_{R_j}^T m\delta\phi_{I_k}; \quad j \neq k \end{aligned} \qquad (11b)$$

which are expressions for the diagonal and off-diagonal elements of C. Since $\sigma_j = -\zeta_j\omega_{0_j}$, it follows that the diagonal elements of C correspond to modal damping as in the case of proportional damping. The off-diagonal elements of C can be calculated provided that $\delta\phi_I$ can be measured.

One possible way to measure $\delta\phi_I$ suggests itself from the co-quad technique currently employed to measure modal frequencies and damping.[4,5] The acceleration response of a given point on a structure excited in one of its "normal" modes is separated into coincident and quadrature components (co and quad) with respect to either the driving force or some other response point on the structure. These components are 90° out of phase with each other, the coincident part defined to be in phase with the reference function. They are thus representable as real and imaginary components in the complex plane as illustrated by Fig. 2. In the case where the driving force is taken to be the reference function, the real part of the modal displacement corresponds to the quadrature component of response since response is nearly 90° out of phase with the excitation at resonance. The coincident response data are not fully utilized. The method proposed herein could make use of this information to construct the off-diagonal terms of the modal damping matrix, provided that sufficiently "pure" modes are obtainable.

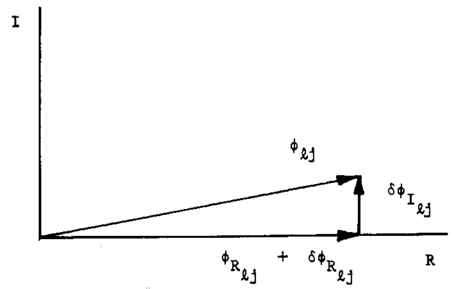

Fig. 2 Vectorial representation of the lth element of the jth eigenvector in the complex plane.

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Control Theory Formulation for Nonlinear Elastic Analysis of Trusses

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Introduction

METHODS for the analysis of structures for which the application of the superposition principles is not justified normally envisage the solution of a set of nonlinear equations (either algebraic or differential).[1,2,4] It is desirable to develop methods of nonlinear structural analysis which result in a set of nonlinear equations with a smaller number of basic unknowns. To do this, herein, the structure is visualized as a

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